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Strong Stability of Impulsive Systems

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The strong stability of the zero solution of impulsive systems with impulses at fixed moments of time is investigated. It is proved that the existence of piecewise continuous functions with certain properties is a necessary and sufficient condition for the strong stability of the zero solution of such systems. By using differential inequalities for piecewise continuous functions, sufficient conditions for the strong stability of the zero solution are found.

1. INTRODUCTION

Systems of impulsive differential equations have many applications in various fields of science and technology. By means of these systems one can simulate processes and phenomena subject during their evolution to short-time perturbations of negligible duration. These perturbations lead to changes by jumps of the state of the system. Such systems have attracted much attention recently (Leela, 1977a,b; Pandit, 1977; Pavlidis, 1967; Rao and Rao, 1977; Myshkis and Samoilenko, 1967; Samoilenko and Perestjuk, 1977; Simeonev and Bainov, 1985, 1986; Dishliev and Bainov, 1985a,b).

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the norm $\|x\| = (\sum_1^n x_i^2)^{1/2}$. Let $I = [0, \infty)$, $\Gamma = \{(t, x) \in I \times \mathbb{R}^n\colon \|x\| < h\} = I \times B_h$, $0 < h =$ const, $B_h = \{x \in \mathbb{R}^n\colon \|x\| < h\}$.

Consider the system of impulsive differential equations in fixed moments of time

$$\begin{aligned} \dot{x} &= f(t, x), \qquad t \neq t_i \\ \Delta x|_{t=t_i} &= I_i(x(t_i)) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $f\colon \Gamma \to \mathbb{R}^n$, $\Delta x|_{t=t_i} = x(t_i + 0) - x(t_i - 0)$, and the impulse moments $\{t_i\}$ form a strictly increasing sequence, i.e., $0 < t_1 < t_2 < \cdots$ and $\lim_{i\to\infty} t_i = \infty$.

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Impulse systems of the form (1) are characterized in the following way: For $t \in I$, $t \neq t_i$, the solution of system (1) is determined by system $\dot{x} = f(t, x)$. At the moments $t = t_i$ the mapping point $(t, x(t))$ under the influence of a brief force (impact, impulse) is transferred momentarily from the position $(t_i, x(t_i))$ to the position $(t_i, x(t_i) + I_i(x(t_i)))$. The solutions of system (1) are piecewise continuous functions with points of discontinuity of first type $\{t_i\}$ in which it is left continuous, i.e., the following relations hold:

$$x(t_i - 0) = x(t_i), \qquad x(t_i + 0) = x(t_i) + \Delta x(t_i) = x(t_i) + I_i(x(t_i))$$

In the present paper the strong stability of the zero solution of system (1) is investigated. In these investigations piecewise continuous auxiliary functions are used, which are analogs to Liapunov's functions (Simeonov and Bainov, 1986), as well as differential inequalities for piecewise continuous functions.

2. PRELIMINARY NOTES AND DEFINITIONS

We shall say that conditions (A) are satisfied if the following conditions hold:

A1. The function $f(t, x)$ is continuous in Γ and has continuous partial derivatives of first order with respect to all components of x.

A2. $f(t, 0) = 0$ for $t \in I$.

A3. The functions $I_i(x)$, $i = 1, 2, \ldots$, are continuously differentiable in B_h and $I_i(0) = 0$.

A4. There exists a number μ $(0 < \mu < h)$ such that if $x \in B_\mu$, then $x + I_i(x) \in B_h$, $i = 1, 2, \ldots$.

A5. The functions $J_i(x) = x + I_i(x)$, $i = 1, 2, \ldots$, are invertible in B_h and $J_i^{-1}(x) \in B_h$ for $x \in B_h$.

A6. The impulse moments $\{t_i\}$ form a strictly increasing sequence, which tends to infinity for $i \to \infty$.

If conditions (A) are satisfied, then for each point $(t_0, x_0) \in \Gamma$ there exists a unique solution $x(t)$, $t > t_0$, of system (1) satisfying the initial condition $x(t_0 + 0) = x_0$. We shall denote this solution by $x(t) = x(t; t_0, x_0)$ and let $\mathscr{I}^+(t_0, x_0)$ be the maximal interval of the form (t_0, ω) in which $x(t; t_0, x_0)$ is defined.

We shall say that condition (B) is satisfied if the following condition holds:

B. Each solution $x(t; t_0, x_0)$ of system (1) that satisfies the estimate

$$\|x(t; t_0, x_0)\| \leq h_1 < h \qquad \text{for} \quad t \in \mathscr{I}^+(t_0, x_0)$$

is defined for $t > t_0$.

Consider the sets

$$G_i = \{(t, x) \in \Gamma: t_{i-1} < t < t_i\}, \qquad i = 1, 2, \ldots$$

$$S_\alpha = \{(t, x) \in \Gamma: x \in B_\alpha \text{ if } (t, x) \in \bigcup_1^\infty G_i \text{ and } x + I_i(x) \in B_\alpha$$

$$\text{if } t = t_i\}, \qquad 0 < \alpha = \text{const}$$

We shall give definitions of stability of the zero solution of system (1) that correspond to those in Samoilenko and Perestjuk (1977) and are of the form used in Rouche *et al.* (1977).

Definition 1. The zero solution of system (1) is called:

(a) Stable if $(\forall \varepsilon > 0)$ $(\forall t_0 \in I)$ $(\exists \delta = \delta(t_0, \varepsilon) > 0)$ $(\forall x_0 \in B_h: (t_0, x_0) \in S_\delta)$ $(\forall t \in \mathcal{J}^+(t_0, x_0))$: $\|x(t; t_0, x_0)\| < \varepsilon$.

(b) Uniformly stable if the number δ from (a) does not depend on t_0.

(c) Asymptotically stable if it is stable and $(\forall t_0 \in I)$ $(\exists \lambda = \lambda(t_0) > 0)$ $(\forall x_0 \in B_h: (t_0, x_0) \in S_\lambda)$: $\lim_{t \to \infty} x(t; t_0, x_0) = 0$.

Definition 2. The zero solution of system (1) is called strongly stable if $(\forall \varepsilon > 0)$ $(\exists \delta = \delta(\varepsilon) > 0)$ $(\forall t_0 \in I)$ $(\forall x_0 \in B_h: (t_0, x_0) \in S_\delta)$ $(\forall t \in I)$: $\|x(t; t_0, x_0)\| < \varepsilon$.

It is clear that from the strong stability of the zero solution its uniform stability follows, which in turn implies its stability. Moreover, strong stability and asymptotic stability are always independent.

We introduce the classes $\mathcal{V}_0$, $\mathcal{V}_1$ and $\mathcal{V}_2$ of piecewise continuous functions.

Definition 3. We shall say that the function $V: \Gamma \to \mathbb{R}$ belongs to the class $\mathcal{V}_0$ if V is continuous in each of the sets G_i, $V(t, 0) = 0$ for $t \in I$, and for each $i = 1, 2, \ldots$ and each point $x_0 \in B_h$ there exist and are finite the limits

$$V(t_i - 0, x_0) = \lim_{\substack{(t,x) \to (t_i, x_0) \\ t < t_i}} V(t, x), \qquad V(t_i + 0, x_0) = \lim_{\substack{(t,x) \to (t_i, x_0) \\ t > t_i}} V(t, x)$$

and the equality $V(t_i - 0, x_0) = V(t_i, x_0)$ holds.

Definition 4. We shall say that the function $V: \Gamma \to \mathbb{R}$ belongs to class $\mathcal{V}_1$ if $V \in \mathcal{V}_0$ and V is continuously differentiable in each of the sets G_i.

For each function $V \in \mathcal{V}_1$ we define the function

$$\dot{V}(t, x) = \frac{\partial V(t, x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial V(t, x)}{\partial x_i} f_i(t, x) \tag{2}$$

for $(t, x) \in \bigcup_1^\infty G_i$.

If $x(t)$ is a solution of system (1), then

$$\frac{d}{dt} V(t, x(t)) = \dot{V}(t, x()t)) \qquad \text{for} \quad t \in I, \quad t \neq t_i \tag{3}$$

Definition 5. We shall say that the function $V : \Gamma \to \mathbb{R}$ belongs to class $\mathcal{V}_2$ if $V \in \mathcal{V}_1$ and V has continuous partial derivatives of second order in each of the sets G_i.

Let $V \in \mathcal{V}_2$. If $f(t, x)$ satisfies condition A1 and has a continuous partial derivative with respect to t, we can define the function

$$\ddot{V}(t, x) = \frac{\partial \dot{V}(t, x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial \dot{V}(t, x)}{\partial x_i} f_i(t, x) \tag{4}$$

for $(t, x) \in \bigcup_1^\infty G_i$.

In the further considerations we shall use the class K of all functions $a(r) : I \to I$ that are continuous and strictly increasing and such that $a(0) = 0$.

3. MAIN RESULTS

3.1. Liapunov's Functions and Strong Stability

We shall establish that the existence of functions of class $\mathcal{V}_1$ with certain properties is a necessary and sufficient condition for the strong stability of the zero solution of system (1).

Theorem 1. Let conditions (A) and (B) hold.

Then the zero solution of system (1) is strongly stable if and only if there exists a function $V \in \mathcal{V}_1$ with the following properties:

(i) $a(\|x\|) \leq V(t, x) \leq b(\|x\|)$ for $(t, x) \in \Gamma$ and $a, b \in K$.

(ii) $\dot{V}(t, x) \equiv 0$ for $(t, x) \in \Gamma$, $t \neq t_i$.

(iii) $V(t_i + 0, x + I_i(x)) = V(t_i, x)$, $i = 1, 2, \ldots$, $x \in B_h$.

Proof. Sufficiency. Let $\varepsilon > 0$ be given and let

$$\delta = \delta(\varepsilon) < \min\{\varepsilon, b^{-1}(a(\varepsilon)), b^{-1}(a(\mu))\}$$

If $x(t) = x(t; t_0, x_0)$ is a solution of system (1) such that $t_0 \in I$ and $(t_0, x_0) \in S_\delta$, then from (3) and the conditions of Theorem 1 we deduce that for $t \in I$ the following inequalities hold:

$$a(\|x(t)\|) \leq V(t, x(t)) = V(t_0 + 0, x_0) \leq b(\|x_0\|)$$
$$< b(\delta) < \min\{a(\varepsilon), a(\mu)\}$$

Hence $\|x(t; t_0, x_0)\| < \min\{\varepsilon, \mu\}$ for $t \in I$, i.e. the zero solution of system (1) is strongly stable.

Necessary. Let the zero solution of system (1) be strongly stable. We shall prove that the function

$$V(t,x)=\begin{cases}\|x(0;\,t,x)\| & \text{for } (t,x)\in\bigcup_1^\infty G_i\\ V(t_i-0,x) & \text{for } t=t_i\end{cases}$$

belongs to class $\mathscr{V}_1$ and satisfies conditions (i)-(iii).

The continuity of the partial derivatives of $V(t,x)$ in each of the sets G_i follows from the smoothness of the functions $f(t,x)$ and $I_i(x)$, $i=1,2,\ldots$.

It is also clear that $V(t,0)=0$, $t\in I$.

Let $i\in\mathbb{N}$ and $x_0\in B_h$. Then $V(t_i-0,x_0)=V(t_i,x_0)$. Moreover,

$$V(t_i+0,x_0)=\lim_{\substack{(t,x)\to(t_i,x_0)\\ t>t_i}}V(t,x)=\|x(0;\,t_i,x_0+I_i(x_0))\|$$

$$=V(t_i,x_0+I_i(x_0))$$

Hence the function $V(t,x)$ belongs to class $\mathscr{V}_1$.

Let $x(0;\,t,x)=z$. Then $x=x(t;\,0,z)$. From the strong stability of the zero solution of system (1) it follows that $\|x\|=\|x(t;\,0,z)\|\le\delta(\|z\|)$ for $t\in I$, where $\delta(\varepsilon)$ is the number corresponding to ε in Definition 2. We can assume that $\delta(\varepsilon)$ is a continuous and strictly increasing function of ε. Then $V(t,x)=\|z\|\ge\varepsilon(\|x\|)$, where $\varepsilon(\delta)$ is the function inverse to $\delta(\varepsilon)$. It is clear that $\varepsilon(\delta)\in K$.

On the other hand, if $\bar t\in I$ and $\|\bar x\|<\delta(\varepsilon)$, then $\|x(t;\,\bar t,\bar x)\|<\varepsilon$ for all $t\in I$. Hence $V(\bar t,\bar x)=\|x(0;\,\bar t,\bar x)\|<\varepsilon$, which shows that $V(t,x)\to 0$ for $x\to 0$ uniformly in I. Hence there exists a function $b\in K$ sich that $V(t,x)\le b(\|x\|)$ for $(t,x)\in\Gamma$, which proves that the function $V(t,x)$ satisfies condition (i).

Let $t\ne t_i$ $(i=1,2,\ldots)$. Since

$$V(t,x(t;\,t_0,x_0))=\|x(0;\,t,x(t;\,t_0,x_0))\|=\|x(0;\,t_0,x_0)\|$$

then $V(t,x(t;\,t_0,x_0))$ does not depend on t. Hence $\dot V(t,x)\equiv 0$ for $(t,x)\in\bigcup_1^\infty G_i$, i.e., the function $V(t,x)$ satisfies condition (ii).

From the equalities

$$x(0;\,t_i,x+I_i(x))=x(0;\,t_i+0,x+I_i(x))=x(0;\,t_i-0,x)$$

it follows that $V(t_i+0,x+I_i(x))=V(t_i-0,x)=V(t_i,x)$, i.e., condition (iii) is satisfied. This completes the proof of Theorem 1. ■

3.2. Differential Inequalities and Strong Stability

We shall prove two lemmas which generalize well-known results on differential inequalities. For this purpose we consider the following scalar equations with impulses:

$$\dot{u} = \omega_1(t, u), \quad t \neq t_i; \qquad \Delta u|_{t=t_i} = P_i(u(t_i)) \tag{5}$$

where $\omega_1 : [t_0 - T, t_0] \times \Omega \to \mathbb{R}$, Ω is an open interval in $\mathbb{R}$, and t_0 and T are positive constants, $t_0 > T$, $P_i : \Omega \to \Omega$;

$$\dot{v} = \omega_2(t, v), \quad t \neq t_i; \qquad \Delta v|_{t=t_i} = P_i(v(t_i)) \tag{6}$$

where $\omega_2 : [t_0, t_0 + T] \times \Omega \to \mathbb{R}$;

$$\begin{aligned} \ddot{u} &= \omega(t, u, \dot{u}), \qquad t \neq t_i \\ \Delta u|_{t=t_i} &= A_i(\dot{u}(t_i)) \\ \Delta \dot{u}|_{t=t_i} &= B_i(u(t_i), \dot{u}(t_i)) \end{aligned} \tag{7}$$

where $\omega : [t_0 - T, t_0 + T] \times \Omega_1 \times \Omega_2 \to \mathbb{R}$, $A_i : \Omega_2 \to \Omega_1$, $B_i : \Omega_1 \times \Omega_2 \to \Omega_2$, Ω_1 and Ω_2 are open intervals in $\mathbb{R}$.

Consider as well the impulsive system

$$\begin{aligned} \dot{u} &= F(t, u), \qquad t \neq t_i \\ \Delta u|_{t=t_i} &= C_i(u(t_i)) \end{aligned} \tag{8}$$

where $u \in \mathbb{R}^m$, $F : I \times \mathbb{R}^m \to \mathbb{R}^m$.

Lemma 1. Let the following conditions hold:

1. The function $w_1(t, u)$ is continuous in $[t_0 - T, t_0] \times \Omega$ and $u^*(t) : [t_0 - T, t_0] \to \Omega$ is the maximal solution of (5) such that $u(t_0 + 0) = u \in \Omega$.

2. The function $\omega_2(t, v)$ is continuous in $[t_0, t_0 + T] \times \Omega$ and $v^*(t)$: $[t_0, t_0 + T] \to \Omega$ is the maximal solution of (6) such that $v^*(t_0 + 0) = u_0$.

3. The functions $u + P_i(u)$, $i = 1, 2, \ldots$, are monotone increasing in Ω.

4. The function $w(t) : [t_0 - T, t_0 + T] \to \Omega$ is continuously differentiable in each of the intervals $(t_i, t_{i+1}) \cap [t_0 - T, t_0 + T]$ and in the points $\{t_i\}$ it is left continuous.

5. The following inequalities hold:

$$w(t_0 + 0) \leq u_0 \tag{9}$$

$$\dot{w}(t) \geq \omega_1(t, w(t)), \qquad t \in [t_0 - T, t_0], \quad t \neq t_i \tag{10}$$

$$\dot{w}(t) \leq \omega_2(t, w(t)), \qquad t \in (t_0, t_0 + T], \quad t \neq t_i \tag{11}$$

$$w(t_i + 0) = w(t_i) + P_i(w(t_i)) \tag{12}$$

Then the following inequalities hold:

$$w(t) \leq u^*(t) \qquad \text{for} \quad t \in [t_0 - T, t_0] \tag{13}$$

$$w(t) \leq v^*(t) \qquad \text{for} \quad t \in [t_0, t_0 + T] \tag{14}$$

Proof. Let, for the sake of definiteness, $t_0 \in (t_{k-1}, t_k]$. If $t \in (t_{k-1}, t_0]$, then from (9) and (10), applying Lemma 2.6.ix of Hatvani (1975), we obtain $w(t) \leq u^*(t)$, whence it follows that $w(t_{k-1} + 0) \leq u^*(t_{k-1} + 0)$. Using (12) and (5), we obtain that

$$w(t_{k-1})+P_{k-1}(w(t_{k-1}))\leq u^*(t_{k-1})+P_{k-1}(u^*(t_{k-1}))$$

From this inequality and condition 3 of Lemma 1 it follows that $w(t_{k-1})\leq u^*(t_{k-1})$ and, applying once more Lemma 2.1 of Hatvani (1975), we obtain that $w(t)\leq u^*(t)$ for $t\in(t_{k-2}, t_{k-1}]$. By induction we prove that inequality (12) is satisfied for $t\in[t_0-T, t_0]$. In the same way inequality (14) is proved. ■

For the formulation of the next lemma it is necessary to introduce a partial ordering in $\mathbb{R}^m$, which we shall define in the following natural way.

Definition 6. Let $u, v\in\mathbb{R}^m$. We set $u\geq v$ ($u>v$) if and only if $u_i\geq v_i$ ($u_i>v_i$) for $i=1,2,\ldots,m$, where $u_i(v_i)$ is the ith component of the vector u (v).

Definition 7. We say that the function $F: I\times\mathbb{R}^m\to\mathbb{R}^m$ is quasimonotone increasing if for any two points (t, u), $(t, v)\in I\times\mathbb{R}^m$ and any $i=1,2,\ldots,m$ we have $F_i(t, u)\geq F_i(t, v)$ always when $u_i=v_i$ and $u\geq v$, i.e., for all $t\in I$ and any $i=1,2,\ldots,m$, the function $F_i(t, u)$ is nondecreasing with respect to $(u_1,\ldots,u_{i-1}, u_{i+1},\ldots,u_m)$.

Lemma 2. Let the following conditions be fulfilled:

1. The function $F(t, u)$ is continuous and quasimonotone increasing in $I\times\mathbb{R}^m$ and $u^*(t):[t_0-T, t_0+T]\to\mathbb{R}^m$ is the maximal solution of system (8) such that $u(t_0+0)=u_0\in\mathbb{R}^m$.
2. The functions $u+C_i(u)$ are monotone increasing in $\mathbb{R}^m$.
3. The function $w(t):[t_0-T, t_0+T]\to\mathbb{R}^m$ is continuously differentiable in each of the intervals $(t_i, t_{i+1})\cap[t_0-T, t_0+T]$ and in the points $\{t_i\}$ it is left continuous.
4. The following inequalities hold:

$$w(t_0+0)\leq u_0 \tag{15}$$

$$\dot{w}(t)\leq F(t, w(t)),\qquad t\in[t_0-T, t_0+T],\quad t\neq t_i \tag{16}$$

$$w(t_i+0)=w(t_i)+C_i(w(t_i)) \tag{17}$$

Then

$$w(t)\leq u^*(t)\qquad\text{for}\quad t\in[t_0-T, t_0+T] \tag{18}$$

The proof of Lemma 2 is analogous to the proof of Lemma 1.

Corollary 1. Let the following conditions hold:

1. The function $\omega(t, u_1, u_2):[t_0-T, t_0+T]\times\Omega_1\times\Omega_2\to\mathbb{R}$ is continuous and monotone increasing with respect to u_1 and $u^*(t):[t_0-T, t_0+T]\to\mathbb{R}$ is the maximal solution of equation (7) such that $u^*(t_0+0)=u_0\in\Omega_1$, $\dot{u}^*(t_0+0)=v_0\in\Omega_2$.

2. The inequalities

$$u_1 + A_i(v_1) \le u_2 + A_i(v_2) \tag{19}$$

$$v_1 + B_i(u_1, v_1) \le v_2 + B_i(u_2, v_2) \tag{20}$$

always hold when $u_1 \le u_2$, $v_1 \le v_2$, $u_1, u_2 \in \Omega_1$, $v_1, v_2 \in \Omega_2$.

3. The function $w(t): [t_0 - T, t_0 + T] \to \mathbb{R}$ is twice continuously differentiable in each of the intervals $(t_i, t_{i+1}) \cap [t_0 - T, t_0 + T]$, and in the points $\{t_i\}$ it is left continuious and has a derivative that is left continuous in these points.

4. The following relations hold:

$$w(t_0 + 0) \le u_0$$

$$\dot{w}(t_0 + 0) = v_0$$

$$\ddot{w}(t) \le \omega(t, w(t), \dot{w}(t)), \qquad t \in [t_0 - T, t_0 + T], \quad t \ne t_i$$

$$w(t_i + 0) \le w(t_i) + A_i(\dot{w}(t_i))$$

$$\dot{w}(t_i + 0) \le \dot{w}(t_i) + B_i(w(t_i), \dot{w}(t_i))$$

Then $w(t) \le u^*(t)$ for $t \in [t_0 - T, t_0 + T]$.

The proof of Corollary 1 is a straightforward application of Lemma 2 to the impulsive system

$$\dot{u} = v, \qquad t \ne t_i$$

$$\dot{v} = \omega(t, u, v), \qquad t \ne t_i$$

$$\Delta u|_{t=t_i} = A_i(v(t_i))$$

$$\Delta v|_{t=t_i} = B_i(u(t_i), v(t_i))$$

In the formulation of the main results in this section we shall use the following definition:

Definition 6. The zero solution of system (1) is called uniformly stable to the right (to the left) if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $t_0 \in I$ and $(t_0, x_0) \in S_{\delta(\varepsilon)}$, then $\|x(t; t_0, x_0)\| < \varepsilon$ for all $t > t_0$ (for all $t \in [0, t_0]$).

The notion of uniform stability to the right coincides with the notion of uniform stability introduced by K. P. Persidskii (cf. Definition 1b).

We shall formulate a necessary and sufficient condition for uniform stability to the left of the zero solution of (1).

Lemma 3. The zero solution of system (1) is uniformly stable to the left if and only if for any $\varepsilon > 0$ the following inequality holds:

$$\gamma(\varepsilon) = \inf\{\|x(t; t_0, x_0)\|: t_0 \in I, t \ge t_0, \|x_0\| \ge \varepsilon\} > 0$$

The proof of Lemma 3 is carried out in the same way as the proof of Lemma 2.3 of Hatvani (1975).

If we compare Definition 1 and Definition 6, we obtain the following lemma:

Lemma 4. The zero solution of system (1) is strongly stable if and only if it is stable to the left and to the right at the same time.

Example. Consider the linear impulsive system at fixed moments of time

$$\dot{x} = A(t)x, \quad t \neq t_i; \qquad \Delta x|_{t=t_i} = Q_i x(t_i) \tag{21}$$

where $A(t)$ is a square matrix, the elements of which are continuous functions for $t \in I$; Q_i are constant matrices such that $\det[E + Q_i] \neq 0$ (E is the unit matrix). Let $\phi(t)$ be the fundamental matrix of the system $\dot{x} = A(t)x$ so that $\phi(0) = E$. Introduce the notation $K(t, s) = \phi(t)\phi^{-1}(s)$ for $0 \leq s \leq t < \infty$ and define the operator

$$U(t, s) = \begin{cases} K(t, s) & \text{for} \quad t_i < s \leq t \leq t_{i+1} \\ K(t, t_i)(E + Q_i)K(t_i, s) & \text{for} \quad t_{i-1} < s \leq t_i < t \leq t_{i+1} \\ K(t, t_i)\left[\prod_{j=i}^{k+1} (E + Q_j)K(t_j, t_{j-1})\right](E + Q_k)K(t_k, s) & \\ & \text{for} \quad t_{k-1} < s \leq t_k < t_i < t \leq t_{i+1} \end{cases}$$

Since the nontrivial solution of (21) is given by the formula $x(t; t_0, x_0) = U(t, t_0+0)x_0$, then $x_0 = U^{-1}(t, t_0+0)x(t; t_0, x_0)$. Hence, for any $\varepsilon > 0$ and $\|x_0\| \geq \varepsilon$ we have

$$\varepsilon \leq \|x_0\| \leq \|U^{-1}(t, t_0+0)\| \cdot \|x(t; t_0, x_0)\|$$

whence

$$\|x(t; t_0, x_0)\| \geq \varepsilon \|U^{-1}(t, t_0+0)\|^{-1}$$

On the other hand, for $t = t_0$ and $\|x_0\| = \varepsilon$ we have

$$\|x(t; t_0, x_0)\| = \varepsilon \|U^{-1}(t, t_0+0)\|^{-1}$$

Hence

$$\gamma(\varepsilon) = \inf\{\varepsilon \|U^{-1}(t, t_0+0)\|^{-1} \colon t \geq t_0 \geq 0\}$$

and, applying Lemma 3, we conclude that the zero solution of system (21) is uniformly stable to the left if and only if the function $\|U^{-1}(t, s)\|$ is bounded on the set $0 \leq s \leq t < \infty$. Moreover, it is clear that the zero solution of (21) is uniformly stable to the right if and only if the function $\|U(t, s)\|$ is bounded on the set $0 \leq s \leq t < \infty$. Then, in virtue of Lemma 4, the zero solution of system (21) is strongly stable if and only if the functions $\|\phi(t)\|$ and $\|\phi^{-1}(t)\|$ are bounded for $t \in I$.

We shall formulate and prove the main results in this section.

Theorem 2. Let conditions (A) hold and let there exist a function $V \in \mathscr{V}_1$ satisfying the following conditions:

1. $a(\|x\|) \leq V(t, x) \leq b(\|x\|)$ for $(t, x) \in \Gamma$, where $a, b \in K$.
2. $\omega_1(t, V(t, x)) \leq \dot{V}(t, x) \leq \omega_2(t, V(t, x))$ for $(t, x) \in \Gamma$, $t \neq t_i$, where $\omega_1(t, u), \omega_2(t, u): I \times I \to \mathbb{R}$ are continuous and $\omega_1(t, 0) = \omega_2(t, 0) = 0$ for $t \in I$.
3. The zero solution $u(t) \equiv 0[v(t) \equiv 0]$ of equation (5) [of equation (6)] is uniformly stable to the left (to the right).
4. The functions $u + P_i(u)$ are monotone increasing in I.
5. $V(t_i + 0, x + I_i(x)) = V(t_i, x) + P_i(V(t_i, x))$.

Then the zero solution of system (1) is strongly stable.

Proof. Let $0 < \varepsilon < h$. From condition 3 of Theorem 2 it follows that there exists $\varkappa(\varepsilon) > 0$ such that for $0 \leq \eta < \varkappa(\varepsilon)$ the following inequalities hold:

$$\begin{aligned} &0 \leq u^*(t; t_0, \eta) < a(\varepsilon) \qquad (t_0 \in I, 0 \leq t \leq t_0) \\ &0 \leq v^*(t; t_0, \eta) < a(\varepsilon) \qquad (t \geq t_0) \end{aligned} \tag{22}$$

where $u^*(t; t_0, \eta)$ is the maximal solution of equation (5) such that $u^*(t_0 - 0; t_0, \eta) = \eta$ and $v^*(t; t_0, \eta)$ is the maximal solution of equation (6) such that $v^*(t_0 + 0; t_0, \eta) = \eta$.

Let $0 < \delta(\varepsilon) < \min\{\varepsilon, b^{-1}(\varkappa(\varepsilon))\}$, $t_0 \in I$, $(t_0, x_0) \in S_\delta$, and $x(t) = x(t; t_0, x_0)$ is the solution of (1) such that $x(t_0 + 0; t_0, x_0) = x_0$. From condition 1 of Theorem 2 it follows that $V(t_0 + 0, x_0) < \varkappa(\varepsilon)$. By Lemma 1 and (22) we obtain

$$\begin{aligned} &V(t, x(t)) \leq u^*(t; t_0, V(t_0 + 0, x_0)) < a(\varepsilon) \qquad \text{for} \quad 0 \leq t \leq t_0 \\ &V(t, x(t)) \leq v^*(t; t_0, V(t_0 + 0, x_0)) < a(\varepsilon) \qquad \text{for} \quad t \geq t_0 \end{aligned}$$

i.e., $V(t, x(t)) < a(\varepsilon)$ for all $t \in I$, whence, applying once more condition 1, we obtain that $\|x(t; t_0, x_0)\| < \varepsilon$ for $t \in I$. The last inequality shows that the zero solution of system (1) is strongly stable. ■

Theorem 3. Let the following conditions be fulfilled:

1. Conditions (A) hold.
2. The function $f(t, x)$ has a continuous partial derivative with respect to t.
3. There exists a function $V \in \mathscr{V}_2$ and functions $a, b, c \in K$ such that:

(i) $a(\|x\|) \leq V(t, x) \leq b(\|x\|)$, $(t, x) \in \Gamma$.

(ii) $\dot{V}(t, x) \leq c(\|x\|)$ for $(t, x) \in \bigcup_1^\infty G_i$, i.e., $\dot{V}(t, x) \to 0$ for $x \to 0$ uniformly in I.

(iii) $\ddot{V}(t, x) \leq \omega(t, V(t, x), \dot{V}(t, x))$ for $(t, x) \in \Gamma$, $t \neq t_i$, where $\omega(t, u_1, u_2)$: $I \times I \times \mathbb{R} \to \mathbb{R}$ is continuous and monotone increasing on u_1 and $\omega(t, 0, 0) = 0$ for $t \in I$.

(iv) $V(t_i+0, x+I_i(x)) \leq V(t_i, x) + A_i(\dot{V}(t_i, x))$.

(v) $\dot{V}(t_i+0, x+I_i(x)) \leq \dot{V}(t_i, x) + B_i(V(t_i, x), \dot{V}(t_i, x))$.

4. Condition 2 of Corollary 1 is satisfied.

5. The zero solution of equation (7) is strongly u-stable, i.e., $(\forall \varepsilon > 0)$ $(\exists \delta(\varepsilon) > 0)$ $(\forall t_0 \in I)$ $(\forall u_0\ 0 \leq u_0 < \delta(\varepsilon))$ $(\forall \dot{u}_0 \in \mathbb{R}, |\dot{u}_0| < \delta(\varepsilon))$ $(\forall t \in I)$: $0 \leq u(t;\ t_0, u_0, \dot{u}_0) < \varepsilon$.

Then the zero solution of system (1) is strongly stable.

The proof of Theorem 3 is analogous to the proof of Theorem 2, with the only difference that instead of Lemma 1, Corollary 1 is used.

REFERENCES

Dishliev, A. B., and Bainov, D. D. (1985a). Sufficient conditions for absence of "beating" in systems of differential equations with impulses, *Applicable Analysis*, **18**, 67-73.

Dishliev, A. B., and Bainov, D. D. (1985b). Conditions for the absence of the phenomenon "beating" for systems of impulse differential equations, *Bulletin of the Institute of Mathematics, Academia Sinica*, **13**, 237-256.

Hatvani, L. (1975). On the strong stability by Liapunov's direct method, *Acta Scientarum Mathematicarum*, **37**, 241-251.

Leela, S. (1977a). Stability of measure differential equations, *Pacific Journal of Mathematics*, **55**, 489-498.

Leela, S. (1977b). Stability of differential systems with impulsive perturbations in terms of two measures, *Nonlinear Analysis, Theory, Methods, Applications*, **1**, 667-677.

Myshkis, A. D., and Samoilenko, A. M. (1967). Systems with impulses in prescribed moments of the time, *Math. Sbornik*, **74**, 202-208 (*in Russian*).

Pandit, S. G. (1977). On the stability of impulsively perturbed differential systems, *Bulletin of the Australian Mathematical Society*, **17**, 423-432.

Pavlidis, T. (1967). Stability of systems described by differential equations containing impulses, *IEEE Transactions on Automatic Control*, **2**, 43-45.

Rao, M. R. R., and Rao, V. S. H. (1977). Stability of impulsively perturbed systems, *Bulletin of the Australian Mathematical Society*, **16**, 99-110.

Rouche, N., Habets, P., and Laloy, M. (1977). *Stability Theory by Liapunov's Direct Method*, Springer-Verlag, New York.

Samoilenko, A. M., and Perestjuk, N. A. (1977). Stability of the solutions of differential equations with impulse effect, *Diff. Uravn.*, **11**, 1981-1992 (in Russian).

Simeonov, P. S., and Bainov, D. D. (1985). The second method of Liapunov for systems with impulse effect, *Tamkang Journal of Mathematics*, **16**, 19-40.

Simeonov, P. S., and Bainov, D. D. (1986). Stability with respect to part of the variables in systems with impulse effect, *Journal of Mathematical Analysis and Applications*, **117**, 247-263.